Student Number__________________

ASCHAM SCHOOL

2020

YEAR 12

TRIAL

EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading time – 10 minutes.
- Working time – 3 hours.
- Write using black non-erasable pen.
- NESA-approved calculators may be used.
- A NESA Reference Sheet is provided.
- All necessary working should be shown in every question.

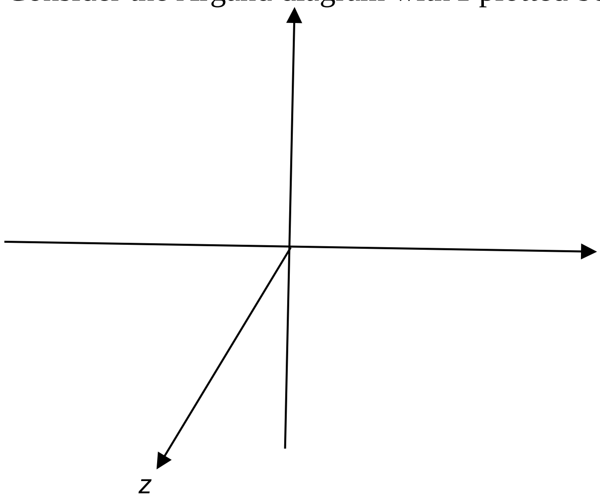
Total marks – 100

- Attempt Sections A and B.
- Section A is worth 10 marks.
- Recommended time on Section A: 15 minutes
- Answer Section A on the multiple choice answer sheet.
- Detach the multiple choice answer sheet from the back of the examination paper.
- Section B contains 6 questions worth 15 marks each.
- Recommended time on Section B: 2 hours 45 minutes
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher.

SECTION A – 10 MULTIPLE CHOICE QUESTIONS 10 MARKS

ANSWER ON THE ANSWER SHEET

1 Consider the Argand diagram with z plotted below.



Which of the following could be the complex number z?

A $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

B $\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)$

C $-\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}$

D $\cos\left(-\frac{2\pi}{3}\right)-i\sin\left(-\frac{2\pi}{3}\right)$

2 Two of the roots of the equation $z^4+Bz^3+Cz^2+Dz+E=0$, where B, C, D, E are real are $3+i$ and $1-2i$. The value of E is:

A – 50

B – 24

C 24

D 50

3 Consider the proposition:

$$P(n): \sum_{k=1}^{n}(2k-1) = n^2 .$$

Which of the following is true for $P(k+1)$?

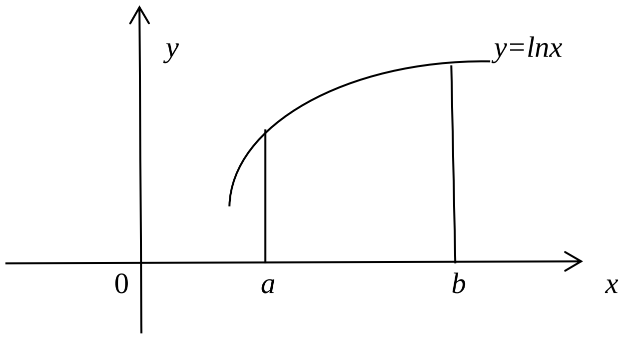
A $(2k-1)+(2k+1)=(k+1)^2$

B $\sum_{1}^{k}(2k-1)+2k=(k+1)^2$

C $\sum_{1}^{k}(2k-1)+2(k+1)=(k+1)^2$

D $\sum_{1}^{k}(2k-1)+(2k+1)=(k+1)^2$

4 If $f(x)=\ln x$ is the continuous, strictly increasing function on the interval $[a,b]$, as shown below, which of the following three statements must be true?



I $\int_a^b \ln x dx < (b-a)\ln b$

II $\int_a^b \ln x dx > (b-a)\ln a$

III there exists a number c where $a < c < b,$ such that $\int_a^b \ln x dx = (b-a)\ln c$

A I only

B II only

C III only

D I, II and III

5 The velocity of a particle at $x=-3$ with simple harmonic motion described by $\ddot{x}=-12(x+3)$ where the amplitude is 4, is:

A 1

B 0

C $\pm\sqrt{192}$

D Insufficient information

6 Which of the following is a counter-example to the following statement?
All people who get ATARS over 99 do Extension 2 Mathematics.

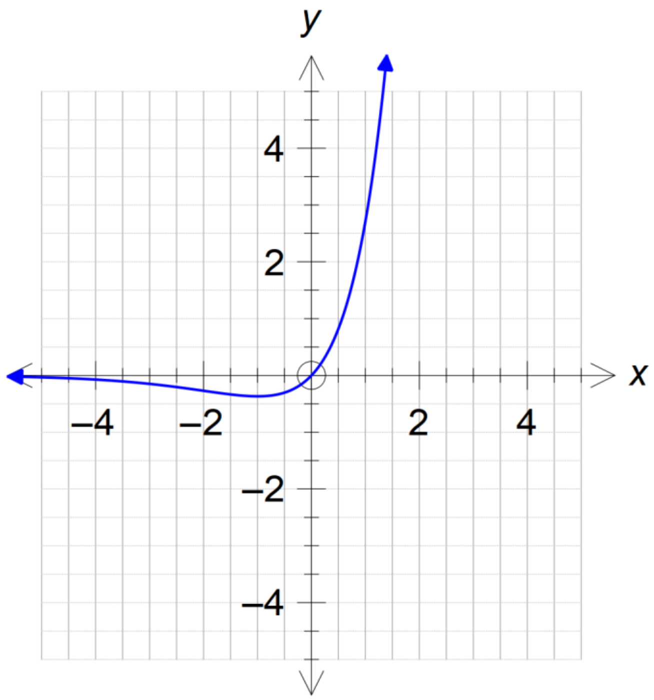
A No people who get ATARS over 99 do Extension 2 Mathematics.

B Jesse did History and got an ATAR over 99.

C Sam did Extension 2 Mathematics and got an ATAR under 99.

D Polly did not do Extension 2 Mathematics and got an ATAR over 99.

7 The equation of the graph below could be:



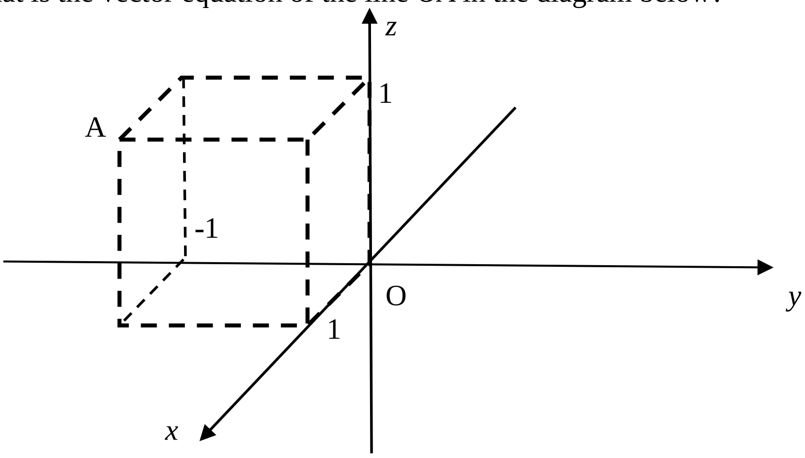
A $y=xe^{x}$

B $y=xe^{-x}$

C $y=x\ln x$

D $y=-x\ln x$

8 What is the vector equation of the line OA in the diagram below?



A $\begin{pmatrix}1\\-1\\1\end{pmatrix}+k\begin{pmatrix}0\\-1\\0\end{pmatrix}$

B $\begin{pmatrix}1\\-1\\1\end{pmatrix}+k\begin{pmatrix}1\\0\\0\end{pmatrix}$

C $\begin{pmatrix}1\\-1\\1\end{pmatrix}+k\begin{pmatrix}0\\0\\1\end{pmatrix}$

D $\begin{pmatrix}1\\-1\\1\end{pmatrix}+k\begin{pmatrix}1\\-1\\1\end{pmatrix}$

9 If $\dfrac{dQ}{dt}=k\left(Q-A\right)$ for $k, A\in\mathbb{R}$, then which of the equations below could be a solution?

A $Q=k\left(\dfrac{Q^2}{2}-AQ\right)$ where $k, A\in\mathbb{R}$

B $Q=Ae^{kt}$ where $k, A\in\mathbb{R}$

C $Q=A+Be^{kt}$ where $k, A, B\in\mathbb{R}$

D $Q=\dfrac{A}{1+Be^{kt}}$ where $k, A, B\in\mathbb{R}$

10 Consider the set of points described by $|z+2|=1$ on the complex plane. What is the maximum value of $\arg z$?

A $\dfrac{\pi}{6}$

B $\dfrac{\pi}{3}$

C $\dfrac{2\pi}{3}$

D $\dfrac{5\pi}{6}$

SECTION 2 – 6 QUESTIONS EACH WORTH 15 MARKS

Question 11 – Begin a new writing booklet

a Find $\int \frac{x^3\,dx}{x^8+3}$. **2**

b Find $\int \cos^{-1} x\;dx$. **2**

c Find $\int \sec^4 x\;dx$. **3**

d

The points *A*, *B*, *C*, *D* representing the complex numbers a, b, c, d form a parallelogram as shown in the diagram. $\angle ADC = \frac{2\pi}{3}$ and $|d-c| = |b-c|$.

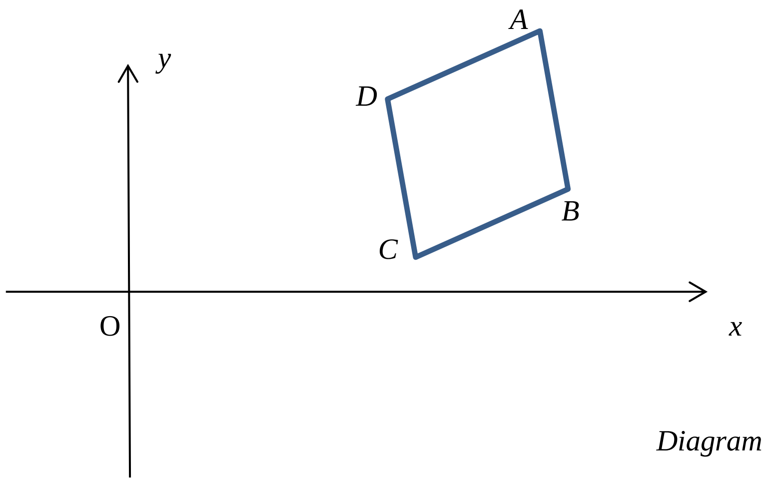


Diagram not to scale

Copy the diagram into your booklet.

i Show that $d-c = (b-c)\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$. **2**

ii Find $\arg\left(\frac{a-d}{b-d}\right)$. **2**

iii Find the value of $\left|\frac{b-a}{d-b}\right|$. **2**

iv Explain why $\arg\left(\frac{a-c}{b-d}\right) = \frac{\pi}{2}$. **2**

Question 12 – Begin a new writing booklet

a Consider the statement for $n \in \mathbb{N}$:

If n is prime then n has exactly two factors.

i Write the converse. **2**

ii Write the contrapositive. **2**

iii Determine whether or not the statement is an equivalence. Give reasons. **2**

iv Georg said that: **2**

If n does not have exactly two factors then n is composite.

Determine whether or not this statement is true. If not, give a counter-example.

b Write the negation of the statement: **2**

All meerkats eat grubs.

c If $x, y \in \mathbb{R}$, solve for *x* and *y*: **3**

$$x + 2y - 3xi + 4yi = 5 + 15i\ .$$

d If $(x + iy)^6 = e^{2i\pi}$, where *x*,*y* are real, find a non-zero solution for *x* and *y*. **2**

Question 13 – Begin a new writing booklet

a The two lines $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix}$ and $\underset{\sim}{q} = \begin{pmatrix} -1 \\ 6 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ intersect at (a,b,c).

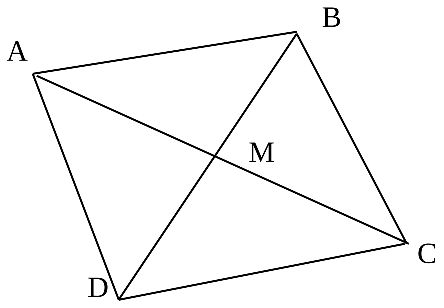
i Find (a,b,c). **2**

ii Show that $\underset{\sim}{r}$ and $\underset{\sim}{q}$ are perpendicular. **2**

iii Find a vector which is perpendicular to both $\underset{\sim}{r}$ and $\underset{\sim}{q}$. **3**

b The point $P(5,7,2)$ lies on the sphere $(x-1)^2 + (y+3)^2 + (z-3)^2 = k^2$. Find the value of *k*. **2**

c Consider the quadrilateral *ABCD* shown. The diagonals *AC* and *BD* bisect each other at *M*. **3**



Copy the diagram.

Use vectors to prove that $\overrightarrow{AB} = \overrightarrow{DC}$.
[Hint: for convenience, let $\overrightarrow{AM} = \underset{\sim}{p}$ and $\overrightarrow{MB} = \underset{\sim}{q}$.]

d Find $\sqrt{15-8i}$. **3**

Question 14 – Begin a new writing booklet

a **i** If *a*, *b* are real, show that $a^2 + b^2 \geq 2ab$. **1**

ii Hence show that if *a*, *b*, *c* are real then $a^2 + b^2 + c^2 \geq ab + bc + ca$. **2**

iii Hence show that $3\left(a^4 + b^4 + c^4\right) \geq \left(a^2 + b^2 + c^2\right)^2$. **2**

b Let $I_n = \dfrac{1}{n!}\int_0^1 x^n\, e^{-x}\, dx,\ n \geq 0$.

i Prove that $\dfrac{1}{n!} = e\left(I_{n-1} - I_n\right)$. **2**

ii Hence evaluate I_4 . **2**

c Find the four 4th roots of -1 and show them on an Argand diagram. **3**

d Prove by mathematical induction that $\forall k \in \mathbb{N}$ and **odd** *n:* **3**

$$\sum_{k=1}^{n} (-1)^{k-1} k^3 = \frac{(2n-1)(n+1)^2}{4}.$$

[Hint: the identity below might be useful:
$2k^3 + 15k^2 + 36k + 27 = (2k+3)(k+3)^2$.]

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Question 15 – Begin a new writing booklet

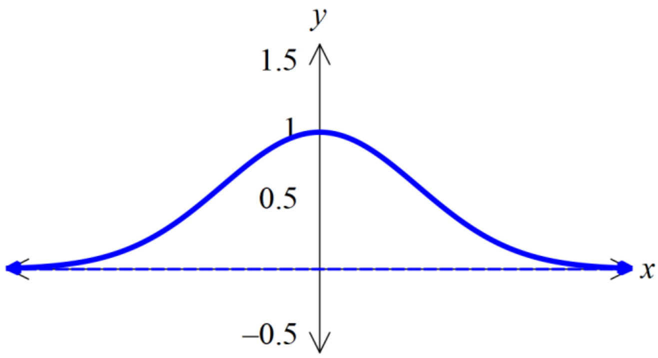
a **i** If $x+\frac{1}{x}=v$ find an expression for $x^3+\frac{1}{x^3}$ in terms of v. **2**

[Hint: expand $\left(x+\frac{1}{x}\right)^3$.]

ii Prove $x^5+\frac{1}{x^5}=v^5-5v^3+5v$. **2**

iii If $x=\cos\theta+i\sin\theta$, using the above parts, find $\cos 10\theta$ in terms of $\cos\theta$. **2**

b For a certain function $f(x)$ the graph $y=f'(x)$ is sketched below.



The graph $y=f(x)$ passes through $(0,0)$. The inverse of $f(x)$ is $g(x)$, that is $f^{-1}(x)=g(x)$.

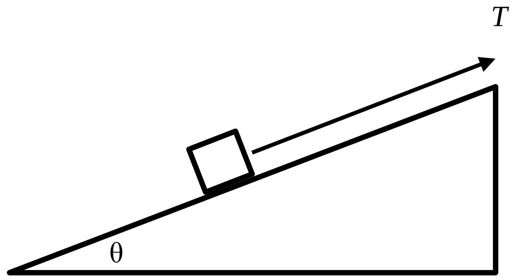
i Sketch $y=f(x)$. **2**

ii Sketch $y=\frac{d}{dx}(g(x))$. **2**

Question 15 continues on the next page…

Question 15 continued…

c A 2 kg mass is being pulled up a slope by a string of tension 10 Newtons. The slope is at an angle θ to the horizontal and the coefficient of friction is 0.3. As well as friction, the forces of gravity *g* and the normal are also acting. **2**



Copy the diagram.

By resolving forces with components along and perpendicular to the slope, find the net force F_{net} in Newtons up the slope.

d Use the substitution $u = \dfrac{1}{x}$ to evaluate $\displaystyle\int_1^{\infty} \frac{dx}{x\sqrt{x^2+2x-1}}$. **3**

Question 16 – Begin a new writing booklet

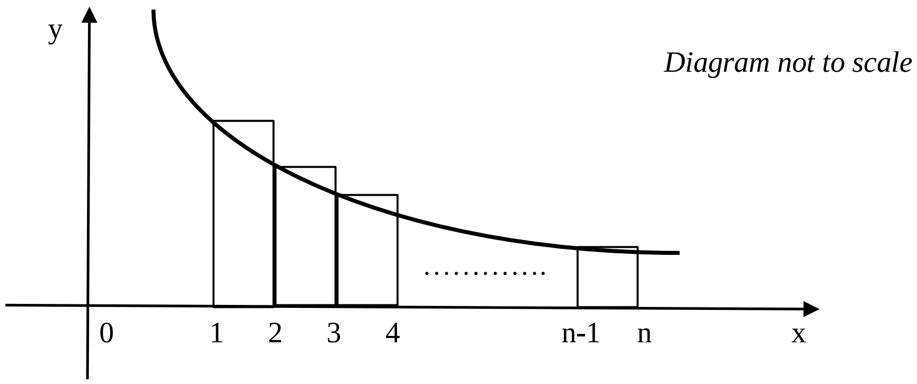
a The Covid-19 1.5 metre rule is based on the premise that a person emits droplets from a height 1.8 m above ground at an angle θ from the horizontal so that the maximum range on the ground is 1.5 metres. Show that the maximum speed *V* m/s of the droplets launched is given by **4**

$$V = \sqrt{\frac{15g}{22}}\,.\qquad \text{(Assume there is no air resistance.)}$$

[Hint: you can assume which angle gives the maximum range…..]

b Solve $\tan^{-1}4x - \tan^{-1}3x = \tan^{-1}\frac{1}{7}$. **2**

c Consider the curve $y = \frac{1}{x}$ sketched below with rectangles above the curve approximating the area under the curve between $x = 1$ and $x = n$.



i Prove that $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n-1} > \ln n$. **4**

ii Does the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ have a limit? Give reasons. **1**

d Let ω be a complex cube root of unity. Prove that if $n \in \mathbb{N}$ then $1 + \omega^n + \omega^{2n} = 3$ if *n* is a multiple of 3 or 0 if *n* is not a multiple of 3. **4**

The end! ☺

Student Number ....

ASCHAM SCHOOL

YEAR 12 Trial Mathematics Extension 2 Exam

MULTIPLE-CHOICE ANSWER SHEET

1.	A ⬭	B ⬭	C ⬭	D ⬭
2.	A ⬭	B ⬭	C ⬭	D ⬭
3.	A ⬭	B ⬭	C ⬭	D ⬭
4.	A ⬭	B ⬭	C ⬭	D ⬭
5.	A ⬭	B ⬭	C ⬭	D ⬭
6.	A ⬭	B ⬭	C ⬭	D ⬭
7.	A ⬭	B ⬭	C ⬭	D ⬭
8.	A ⬭	B ⬭	C ⬭	D ⬭
9.	A ⬭	B ⬭	C ⬭	D ⬭
10.	A ⬭	B ⬭	C ⬭	D ⬭

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Solutions to
Title: 2020 Ascham Math Ext 2 TRIAL Year 12

Q11

a) $\int \frac{x^3\,dx}{x^8+3} = \frac{1}{4}\int \frac{4x^3\,dx}{(x^4)^2+(\sqrt{3})^2}$

(2) $= \frac{1}{4}\times\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^4}{\sqrt{3}}\right)+C$

$= \frac{1}{4\sqrt{3}}\tan^{-1}\left(\frac{x^4}{\sqrt{3}}\right)+C$

b) $\int \cos^{-1}x\,dx$ Let $u=\cos^{-1}x$, $du=\frac{-1}{\sqrt{1-x^2}}dx$, $dv=1dx$, $v=x$

$\int u\,dv = uv-\int v\,du$

$\therefore \int 1\cos^{-1}x\,dx = x\cos^{-1}x-\int\frac{-x\,dx}{\sqrt{1-x^2}}$

$= x\cos^{-1}x+\frac{-1}{2}\int -2x(1-x^2)^{-\frac{1}{2}}dx$

$= x\cos^{-1}x-\frac{1}{2}(1-x^2)^{\frac{1}{2}}\times 2+C$

$= x\cos^{-1}x-\sqrt{1-x^2}+C$ (2)

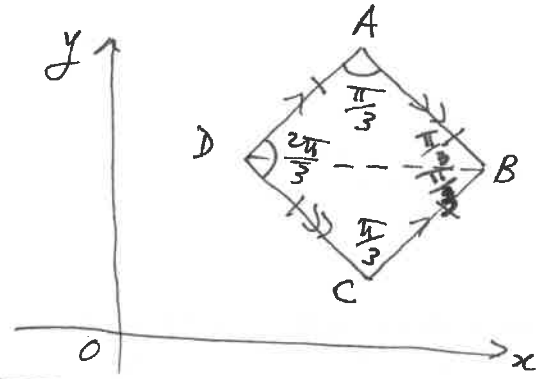
c) $\int \sec^4x\,dx = \int \sec^2x\sec^2x\,dx$

(3) $= \int(\tan^2x+1)\sec^2x\,dx$

$= \int \tan^2x\sec^2x+\sec^2x\,dx$

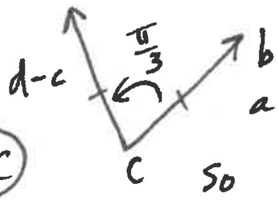
$= \frac{\tan^3x}{3}+\tan x+C$

d)



MC ANSWERS:
1. B 2. D 3. D 4. D 5. C
6. D 7. A 8. D 9. C 10. D

d) i) RTP: $d-c=(b-c)\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$



Since ABCD is a par'm then AB∥DC

So $\angle DCB=\frac{\pi}{3}$ (cointerior ∠s add to π on parallel lines)

Also △DBC is isosceles and DC = CB (given)

∴ △DBC is equilateral since $\angle CDB=\angle CBD=\frac{\pi}{3}$ as well. (∠ sum of △ = π)

$\therefore \arg(d-c)=\arg(b-c)+\frac{\pi}{3}$

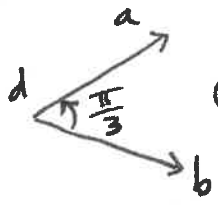
$\therefore d-c=(b-c)\times\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ (2)

(rotate $\frac{\pi}{3}$, same modulus)

$\therefore (d-c)=(b-c)\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$. QED.

ii) $\arg\left(\frac{a-d}{b-d}\right)=\arg(a-d)-\arg(b-d)$

$=\frac{\pi}{3}$ (ABCD is a rhombus since par'm with adjacent sides equal ∴ ∠s bisected by diagonals)



(2)

iii) $\left|\frac{b-a}{d-b}\right|=1$ since △ADB equilateral. (2)

iv) $\arg\left(\frac{a-c}{b-d}\right)=\frac{\pi}{2}$ since ABCD is a rhombus. Diagonals bisect at right angles. (2)

Q12. a) $n \in \mathbb{N}$.

i) If n has exactly 2 factors then n is prime. (2)
$(Q \Rightarrow P)$

ii) If n does not have exactly 2 factors then is is not prime.
$(\neg Q \Rightarrow \neg P)$ (2)

iii) Since $P \Rightarrow Q$ is true and $Q \Rightarrow P$ is true then $P \Leftrightarrow Q$ is true. It is an equivalence. (2)

iv) Statement not true. (2)
Counter-example is $n=1$ has 1 factor but is not composite.

b) At least one meerkat does not eat grubs. (2)

c) $x, y \in \mathbb{R}$:

$x + 2y - 3xi + 4yi = 5 + 15i$

Equate reals & imaginaries:

$x + 2y = 5$ ①

$-3x + 4y = 15$ ②

① ×2 $2x + 4y = 10$

$-(-3x + 4y = 15)$

$5x = -5$

$x = -1$

=> ① $-1 + 2y = 5$ (3)

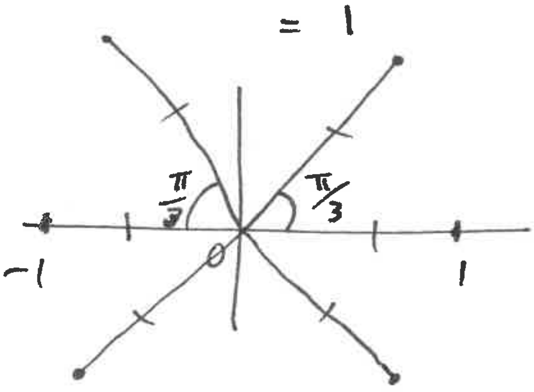
$2y = 6$

$y = 3.$

$\therefore x = -1, y = 3.$

12 d) $(x+iy)^6 = e^{2i\pi}$

$= 1$



A solution could be $1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{1}{2} + \frac{i\sqrt{3}}{2}$

$\therefore x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$.

(anything with mod 1 and arg $\pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$) (roots equally spaced around 1 or −1)

(2)

Q13. a) $\underset{\sim}{r} = \begin{pmatrix}1\\-3\\3\end{pmatrix} + \lambda_1\begin{pmatrix}4\\-5\\2\end{pmatrix}$

$\underset{\sim}{q} = \begin{pmatrix}-1\\6\\7\end{pmatrix} + \lambda_2\begin{pmatrix}1\\2\\3\end{pmatrix}$.

i) $\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}1\\-3\\3\end{pmatrix} + \lambda_1\begin{pmatrix}4\\-5\\2\end{pmatrix}$ AND

$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-1\\6\\7\end{pmatrix} + \lambda_2\begin{pmatrix}1\\2\\3\end{pmatrix}$

$\therefore$ $a = 1 + \lambda_1 4$ & $a = -1 + \lambda_2$

$b = -3 + \lambda_1(-5)$ $b = 6 + 2\lambda_2$

$c = 3 + \lambda_1(2)$ $c = 7 + 3\lambda_2$

$\therefore$ $1 + 4\lambda_1 = -1 + \lambda_2 \Rightarrow \lambda_2 = 2 + 4\lambda_1$

$\Rightarrow b = -3 - 5\lambda_1 = 6 + 2(2 + 4\lambda_1)$

$-3 - 5\lambda_1 = 6 + 4 + 8\lambda_1$

(2) $-13 = 13\lambda_1$

$\therefore \lambda_1 = -1$, $\lambda_2 = 2 + 4(-1) = -2$

$\therefore$ $a = 1 + 4(-1) = -3$

$b = -3 - 5(-1) = 2$ $\therefore \begin{pmatrix}-3\\2\\1\end{pmatrix}$

$c = 3 + 2(-1) = 1$

ii) RTP: $\underset{\sim}{r} \cdot \underset{\sim}{q} = 0$ or directions $= 0$

$\therefore$ $4 \times 1 - 5 \times 2 + 2 \times 3$

$= 4 - 10 + 6$

$= 0$

$\therefore$ Perpendicular. (2)

iii) let $\begin{pmatrix}l\\m\\n\end{pmatrix}$ be perp. to $\underset{\sim}{r}$ & $\underset{\sim}{q}$

$\therefore$ $4l - 5m + 2n = 0$ ① and

$1l + 2m + 3n = 0$. ②

Solve simultaneously:

① × 2 $8l - 10m + 4n = 0$ ③

② × 5 $5l + 10m + 15n = 0$ ④

③ + ④ $13l + 19n = 0$

$\therefore l = \frac{-19n}{13}$

Let $n = 78$, then $l = -19 \times 6 = -114$

and $l + 2m + 3n = 0$

$-114 + 2m + 3 \times 78 = 0$

$2m = -120$ (3)

$m = -60$

$\therefore$ A vector $\perp$ $\underset{\sim}{r}$ & $\underset{\sim}{q}$ is

$\begin{pmatrix}-114\\-60\\78\end{pmatrix}$ or $\begin{pmatrix}19\\10\\-13\end{pmatrix}$

b) $P(5,7,2)$ $(x-1)^2 + (y+3)^2 + (z-3)^2 = k^2$.

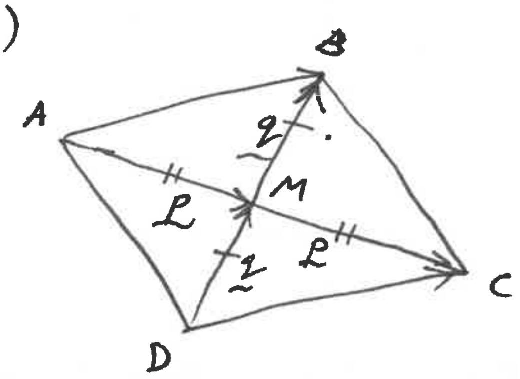
$\Rightarrow (5-1)^2 + (7+3)^2 + (2-3)^2 = k^2$

$4^2 + 10^2 + (-1)^2 = k^2$ (2)

$k^2 = 117$

$k = \sqrt{117}$. $(k > 0)$

c)



RTP: $\overrightarrow{AB} = \overrightarrow{DC}$

Proof: → over page:

Let $\overrightarrow{AM} = \underset{\sim}{l}$ & $\overrightarrow{MB} = \underset{\sim}{q}$

: $\vec{MC} = \underset{\sim}{p}$ and $\vec{DM} = \underset{\sim}{q}$

(given AC + BD bisected at M)

Now in ΔAMB : $\underset{\sim}{p} + \underset{\sim}{q} = \vec{AB}$

In ΔDMC, $\underset{\sim}{q} + \underset{\sim}{p} = \vec{DC}$ (3)

$\therefore \vec{AB} = \vec{DC}$ (both $= \underset{\sim}{p} + \underset{\sim}{q}$) QED.

---

d) Let $\sqrt{15-8i} = a+ib$, $a, b \in \mathbb{R}$

$\therefore 15-8i = (a+ib)^2$

$\therefore 15-8i = a^2 + 2aib - b^2$

$\therefore 15-8i = a^2 - b^2 + 2aib$

Equating: $15 = a^2 - b^2$

$-8 = 2ab$ or $-4 = ab$

By inspection, $a = 4$, $b = -1$

OR $a = -4$, $b = 1$

by convention, $a > 0$ so

$4 - i$ is the root. (3)

Q14 a) i) RTP: $a^2+b^2 \geq 2ab$

Proof: Consider the difference:

$a^2+b^2-2ab = (a-b)^2$ ①

≥ 0 (since square, equality when $a=b$)

$\therefore a^2+b^2 \geq 2ab$.

ii) RTP: $a^2+b^2+c^2 \geq ab+bc+ca$.

Proof: We know that

$a^2+b^2 \geq 2ab$ ②

$b^2+c^2 \geq 2bc$ similarly,

$c^2+a^2 \geq 2ca$ $\forall a,b,c \in \mathbb{R}$

Add: $2a^2+2b^2+2c^2 \geq 2ab+2bc+2ca$

$\therefore 2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

$\therefore a^2+b^2+c^2 \geq ab+bc+ca$

iii) RTP: $3(a^4+b^4+c^4) \geq (a^2+b^2+c^2)^2$.

Proof: Consider the difference:

$3(a^4+b^4+c^4)-(a^2+b^2+c^2)^2$

~~$\geq 3(a^2b^2+b^2c^2+c^2a^2)-(a^2+b^2+c^2)^2$ from (ii)~~

~~$\geq 3a^2b^2+3b^2c^2+3c^2a^2 - (a^2+b^2+c^2)(a^2+b^2+c^2)$~~

~~$\geq 3a^2b^2+3b^2c^2+3c^2a^2 - (a^4+a^2b^2+a^2c^2+b^2a^2+b^4+b^2c^2+c^2a^2+c^2b^2+c^4)$~~

~~$\geq 3a^2b^2+3b^2c^2+3c^2a^2 - (a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2)$~~

~~$\geq a^2b^2+b^2c^2+c^2a^2-a^4-b^4-c^4$~~

$= 3a^4+3b^4+3c^4-(a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2)$

14 a) iii) cont'd

$= 2a^4+2b^4+2c^4-(2a^2b^2+2b^2c^2+2c^2a^2)$

$\geq 2(a^2b^2+b^2c^2+c^2a^2)-2(a^2b^2+b^2c^2+c^2a^2)$ using (ii) ②

≥ 0 as required

$\therefore 3(a^4+b^4+c^4) \geq (a^2+b^2+c^2)^2$ QED!

b) $I_n = \frac{1}{n!}\int_0^1 x^n e^{-x}\,dx,\ n \geq 0$

i) RTP: $\frac{1}{n!} = e(I_{n-1}-I_n)$ ②

Proof: $\int u\,dv = uv - \int v\,du$.

Let $u = x^n$ $\quad dv = e^{-x}dx$

$du = nx^{n-1}dx$ $\quad v = -e^{-x}$

$\therefore \frac{1}{n!}\int_0^1 x^n e^{-x}dx = \frac{1}{n!}\left(uv - \int v\,du\right)$

$= \frac{1}{n!}\left[\left[x^n \cdot -e^{-x}\right]_0^1 - \int_0^1 -e^{-x}\cdot nx^{n-1}dx\right]$

$= \frac{1}{n!}\left[\left(1^n \cdot -e^{-1} - 0\right) + n\int_0^1 x^{n-1}e^{-x}dx\right]$

$= \frac{1}{n!}\left(-\frac{1}{e} + n\int_0^1 x^{n-1}e^{-x}dx\right)$

$\therefore I_n = -\frac{1}{en!} + \frac{1}{(n-1)!}\int_0^1 x^{n-1}e^{-x}dx$

$\therefore +\frac{1}{en!} = I_{n-1} - I_n$

$\therefore \frac{1}{n!} = e(I_{n-1}-I_n)$ QED

ii) $I_0 = \frac{1}{0!}\int_0^1 x^0 e^{-x}dx$

$= \left[-e^{-x}\right]_0^1$

$= -\frac{1}{e}+1$

$I_1 = I_0 - \frac{1}{e \times 1!} = 1-\frac{1}{e}-\frac{1}{e} = 1-\frac{2}{e}$

$I_2 = I_1 - \frac{1}{e\times 2!} = 1-\frac{2}{e}-\frac{1}{2e} = 1-\frac{5}{2e}$

$I_3 = I_2 - \frac{1}{e\times 3!} = 1-\frac{5}{2e}-\frac{1}{6e} = 1-\frac{16}{6e}$

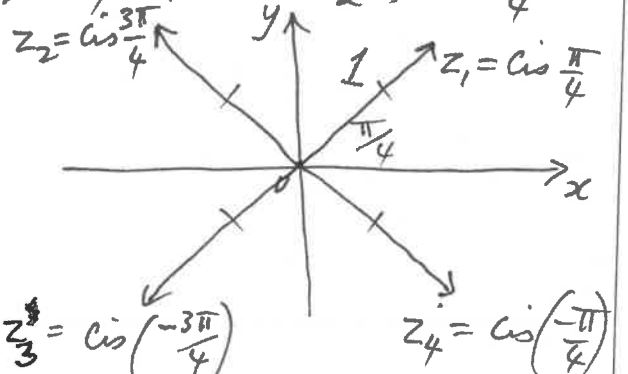
$I_4 = I_3 - \frac{1}{e\times 4!} = 1-\frac{8}{3e}-\frac{1}{24e} = 1-\frac{65}{24e}$

Q14 cont'd

c) Solve $z^4 = -1$. (3)

ie. $(\cos\theta + i\sin\theta)^4 = -1$

$\therefore$ First $\theta = \frac{\pi}{4}$ then equally spaced $\frac{\pi}{2}$ from $\frac{\pi}{4}$.



d) RTP: $\sum_{k=1}^{n} (-1)^{k-1} k^3 = \frac{(2n-1)(n+1)^2}{4}$
ODD n.

Proof: Let $P(n)$ be the proposition that

$1^3 - 2^3 + 3^3 - 4^3 + \ldots + n^3 = \frac{(2n-1)(n+1)^2}{4}$

Prove $P(1)$ true:

$LHS = (-1)^{1-1} 1^3 = 1$

$RHS = \frac{(2(1)-1)(1+1)^2}{4} = \frac{(1)(2)^2}{4} = 1$

$\therefore P(1)$ true.

Assume $P(k)$ true for some odd $k \in \mathbb{N}$:

$1^3 - 2^3 + 3^3 - 4^3 + \ldots + k^3 = \frac{(2(k)-1)(k+1)^2}{4}$

RTP: $P(k+2)$ true ie:

$1^3 - 2^3 + 3^3 - 4^3 + \ldots + k^3 - (k+1)^3 + (k+2)^3$
$= \frac{(2(k+2)-1)(k+2+1)^2}{4}$
$= \frac{(2k+3)(k+3)^2}{4}$

Proof: Consider the LHS of $P(k+2)$:

$\underbrace{1^3 - 2^3 + 3^3 - 4^3 + \ldots + k^3} - (k+1)^3 + (k+2)^3$

$= \frac{(2k-1)(k+1)^2}{4} - (k+1)^3 + (k+2)^3$

$= \frac{(2k-1)(k+1)^2}{4} - \frac{4(k+1)^3}{4} + \frac{4(k+2)^3}{4}$

$= (k+1)^2\left[\frac{2k-1-4(k+1)}{4}\right] + \frac{4[k^3 + 2.3k^2 + 2^2.3k + 8]}{4}$

$= \frac{(k+1)^2[-2k-5]}{4} + \frac{4(k^3+6k^2+12k+8)}{4}$

$= \frac{-(k^2+2k+1)(2k+5)}{4} + \frac{4(k^3+6k^2+12k+8)}{4}$

$= \frac{-(2k^3+5k^2+4k^2+10k+2k+5)}{4} + \frac{4k^3+24k^2+48k+32}{4}$

$= \frac{2k^3+15k^2+36k+27}{4}$

$= \frac{(2k+3)(k+3)^2}{4}$ [Yes, it's true!]

= RHS of $P(k+2)$.

$\therefore P(n)$ true by Math Induction. (3)

Q15 a) i) $x + \frac{1}{x} = v$.

$\left(x + \frac{1}{x}\right)^3 = x^3 + 3x^2 \cdot \frac{1}{x} + 3x \cdot \frac{1}{x^2} + \frac{1}{x^3}$

$= x^3 + \frac{1}{x^3} + 3x + \frac{3}{x}$.

$\therefore x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3x - \frac{3}{x}$

(2) $= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right)$

$= v^3 - 3v$.

ii) $x^5 + \frac{1}{x^5}$? Consider $\left(x + \frac{1}{x}\right)^5$

$= x^5 + 5x^4 \cdot \frac{1}{x} + 10x^3 \cdot \frac{1}{x^2} + 10x^2 \cdot \frac{1}{x^3} + 5x \cdot \frac{1}{x^4} + \frac{1}{x^5}$.

$= x^5 + 5x^3 + 5/x^3 + 10x + \frac{10}{x} + \frac{1}{x^5}$

$= x^5 + \frac{1}{x^5} + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$

$\therefore x^5 + \frac{1}{x^5} = \left(x + \frac{1}{x}\right)^5 - 5\left(x^3 + \frac{1}{x^3}\right) - 10\left(x + \frac{1}{x}\right)$

$= v^5 - 5(v^3 - 3v) - 10v$

$= v^5 - 5v^3 + 15v - 10v$

$= v^5 - 5v^3 + 5v$ (2)

iii) If $x = \cos\theta + i\sin\theta$ then

$x + \frac{1}{x} = \cos\theta + i\sin\theta + \frac{1}{\cos\theta + i\sin\theta}$

$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$\therefore v = 2\cos\theta$.

Now $x^5 + \frac{1}{x^5} = \cos 5\theta + i\sin 5\theta + \cos 5\theta - i\sin 5\theta$

$= 2\cos 5\theta$

$\left(x^5 + \frac{1}{x^5}\right)^2 = x^{10} + 2 + \frac{1}{x^{10}}$ (2)

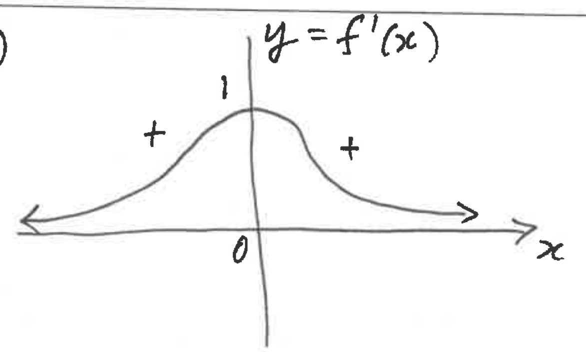
So $x^{10} + \frac{1}{x^{10}} = \left(x^5 + \frac{1}{x^5}\right)^2 - 2$

but $\therefore 2\cos 10\theta = (v^5 - 5v^3 + 5v)^2 - 2$

$= \left[(2\cos\theta)^5 - 5(2\cos\theta)^3 + 5(2\cos\theta)\right]^2 - 2$

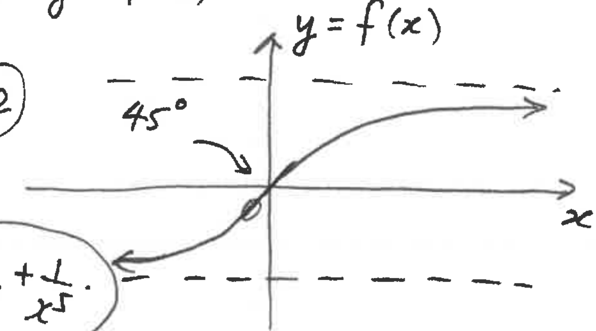
$\therefore \cos 10\theta = \frac{(32\cos^5\theta - 40\cos^3\theta + 10\cos\theta)^2 - 2}{2}$

b) $y = f'(x)$



i) $y = f(x)$

(2)



ii) $g(x) = f^{-1}(x)$.

$y = \frac{d}{dx}(g(x))$ (2)

Now $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$

~~$x = g(y)$ is the same as~~

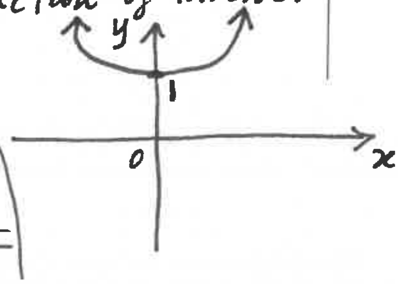
~~$x = f^{-1}(y)$ or $y = f(x)$~~

~~So $\frac{dx}{dy} = g'(y)$~~ Too confusing using algebraic proof unless use dummy variables.

$\therefore \frac{dy}{dx} = \frac{1}{g'(y)}$

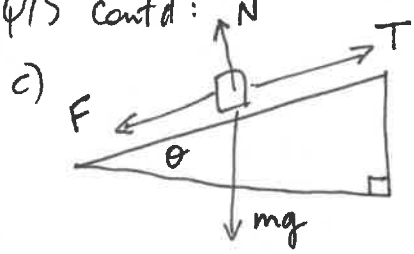
~~$= \frac{1}{g'}$~~

Derivative $\frac{dy}{dx}$ is same as reciprocal function of inverse.



Q15 cont'd:

c)



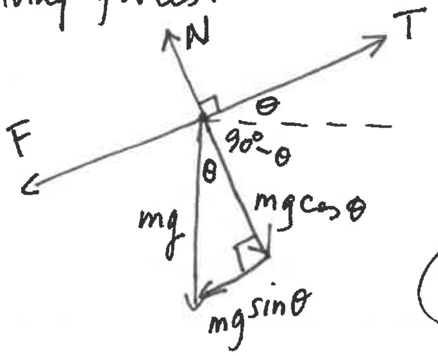
$m = 2kg$

$F = ma$

$T = 10N$

$\mu = 0.3$

Resolving forces:



(2)

Along the slope:

$F_{NET} = T - F - mg\sin\theta$

$= 10 - \mu N - 2g\sin\theta$

$= 10 - 0.3N - 2g\sin\theta$

Perpendicular to slope:

$N = mg\cos\theta = 2g\cos\theta$

$\therefore F_{NET} = 10 - 0.3(2g\cos\theta) - 2g\sin\theta$

$= 10 - 2g(0.3\cos\theta + \sin\theta)$

d) $u = \frac{1}{x} = x^{-1}$

$du = -1x^{-2} = \frac{-1}{x^2}\,dx$

$x = \infty$, $u = 0$

$x = 1$, $u = 1$

$dx = -x^2 du = -\frac{1}{u^2}du$

$$\int_1^\infty \frac{1\,dx}{x\sqrt{x^2+2x-1}}$$

$$= \int_1^0 \frac{u}{\sqrt{\frac{1}{u^2}+\frac{2}{u}-1}}\,dx$$

$$= \int_1^0 \frac{u}{\sqrt{\frac{1+2u-u^2}{u^2}}}\,dx$$

d) cont'd: $u > 0$

$$= \int_1^0 \frac{u}{\frac{1}{u}\sqrt{1+2u-u^2}}\,dx$$

$$= \int_1^0 \frac{u^2}{\sqrt{1-(u^2-2u)}}\,dx$$

$$= \int_1^0 \frac{u^2}{\sqrt{1-(u^2-2u+1-1)}} \times \frac{-1}{u^2}\,du$$

(3)

$$= \int_1^0 \frac{-du}{\sqrt{2-(u-1)^2}}$$

$$= -\left[\sin^{-1}\left(\frac{u-1}{\sqrt{2}}\right)\right]_1^0$$

$$= \left[\sin^{-1}\left(\frac{u-1}{\sqrt{2}}\right)\right]_0^1$$

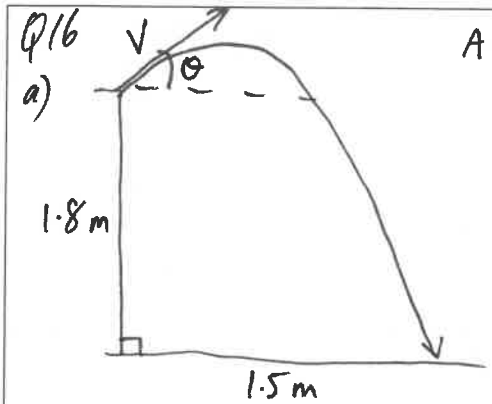
$$= \sin^{-1}\left(\frac{1-1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{0-1}{\sqrt{2}}\right)$$

$$= 0 - -\frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Q16

a)



Assume
$\theta = 45^\circ$
$t=0$
$x=0$
$y=1.8$

$\ddot{x} = 0$ $\quad$ $\ddot{y} = -g$ $\quad$ (4)

$\dot{x} = \int 0\,dt$ $\quad$ $\dot{y} = \int -g\,dt$

$= C_1$ $\quad$ $= -gt + C_2$

When $t=0$, $\dot{x} = V\cos\theta$, $\dot{y} = V\sin\theta$

So $\dot{x} = V\cos\theta = C_1$ $\quad$ $V\sin\theta = 0 + C_2$
$C_2 = V\sin\theta$

$\therefore \dot{x} = V\cos\theta$ $\quad$ $\dot{y} = -gt + V\sin\theta$

$x = \int V\cos\theta\,dt$ $\quad$ $y = \int -gt + V\sin\theta\,dt$

$= Vt\cos\theta + C_3$ $\quad$ $y = \dfrac{-gt^2}{2} + Vt\sin\theta + C_4$

When $t=0$, $x=0$, $y=1.8$

$\therefore 0 = 0 + C_3$, $\quad 1.8 = 0 + 0 + C_4$

$\therefore x = Vt\cos\theta$, $y = \dfrac{-gt^2}{2} + Vt\sin\theta + 1.8$

$\therefore x = Vt \times \dfrac{1}{\sqrt{2}}$, $y = \dfrac{-gt^2}{2} + Vt \times \dfrac{1}{\sqrt{2}} + 1.8$

So $t = \dfrac{x\sqrt{2}}{V}$ and $x = 1.5$, $y = 0$

so $t = \dfrac{1.5\sqrt{2}}{V} \Rightarrow 0 = \dfrac{-g}{2}\left(\dfrac{1.5^2\sqrt{2}^2}{V^2}\right) + V\left(\dfrac{1.5\sqrt{2}}{V\sqrt{2}}\right) + 1.8$

$0 = -g \times \dfrac{9}{4V^2} + 3.3$

$\dfrac{9g}{4V^2} = \dfrac{33}{10} \Rightarrow \dfrac{4V^2}{9g} = \dfrac{10}{33}$

$\therefore V^2 = \dfrac{90g}{33 \times 4} = \dfrac{15g}{22}$ $\quad$ $\therefore |V| = \sqrt{\dfrac{15g}{22}}$

b) $\tan^{-1}4x - \tan^{-1}3x = \tan^{-1}\dfrac{1}{7}$.

Using $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$:

let $A = \tan^{-1}4x$, $B = \tan^{-1}3x$ then

$\therefore \tan A = 4x$, $\tan B = 3x$

So $\tan(\tan^{-1}4x - \tan^{-1}3x)$
$= \tan(\tan^{-1}\frac{1}{7})$

$\therefore \dfrac{4x - 3x}{1 + 4x.3x} = \dfrac{1}{7}$ (one-to-one)

$\dfrac{x}{1+12x^2} = \dfrac{1}{7}$ $\quad$ (2)

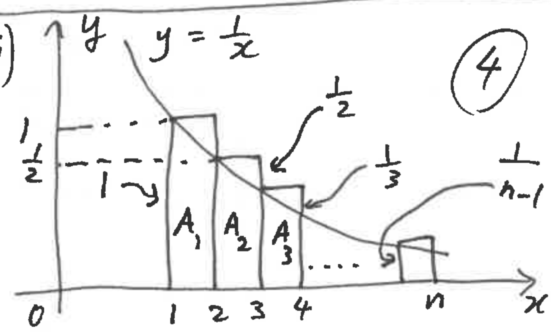
$\therefore 7x = 1 + 12x^2$

$12x^2 - 7x + 1 = 0$

$(4x-1)(3x-1) = 0$

$x = \frac{1}{4}$ or $\frac{1}{3}$.

c) i) $\quad$ (4)



RTP:
$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n-1} > \ln n$.

<u>Proof</u>: Sum of rectangles above curve $> \int_1^n \frac{1}{x}\,dx$

$\therefore A_1 + A_2 + A_3 + \ldots + A_{n-1} > [\ln x]_1^n$

$1 \times 1 + 1 \times \frac{1}{2} + 1 \times \frac{1}{3} + \ldots + 1 \times \frac{1}{n-1} > \ln n - \ln 1$

$\therefore 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} > \ln n$

Q16 cont'd

c) ii) Now $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1} > \ln n$

So since $\ln n \to \infty$ as $n \to \infty$

and $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1} > \ln n$

then $1+\frac{1}{2}+\frac{1}{3}+\ldots \to \infty$ as $n \to \infty$

So No limit. (1)

d) w satisfies $z^3 = 1$. $w \notin \mathbb{R}$.

If $n \in \mathbb{N}$, then RTP:

$1+w^n+w^{2n} = 3$ if $n = 3X$ for some $X \in \mathbb{N}$

OR $1+w^n+w^{2n} = 0$ if $n \neq 3X$

ie $n = 3X+1$ or $n = 3X+2$ in form.

Proof: Consider $n = 3X$ so

$$1+w^n+w^{2n} = 1+w^{3X}+w^{2(3X)}$$
$$= 1+(w^3)^X+(w^3)^{2X}$$

Since $w^3 = 1$ then

$$= 1+1^X+1^{2X}$$
$$= 1+1+1$$
$$= 3 \quad ✓ \quad \text{QED}$$

Now consider $n = 3X+1$ or $n = 3X+2$

So $1+w^n+w^{2n} = 1+w^{3X+1}+w^{2(3X+1)}$

$$= 1+w^{3X}.w^1+w^{6X}.w^2$$
$$= 1+1w+1w^2$$
$$= 0 \quad \text{since } w^3 = 1$$

$[(w^3-1) = (w-1)(w^2+w+1) = 0]$

OR $1+w^n+w^{2n} = 1+w^{3X+2}+w^{2(3X+2)}$

$$= 1+w^{3X}.w^2+w^{6X}.w^4$$
$$= 1+1.w^2+1.w^3.w$$
$$= 1+w^2+1.w$$
$$= 1+w+w^2$$
$$= 0 \quad \text{as well.}$$

(4)

$\therefore 1+w^n+w^{2n} = 3$ if $n = 3X$

or $= 0$ if not.

QED.

Q14 (b) ii) better solution?

$$\frac{1}{n!} = e(I_{n-1} - I_n)$$
$$\frac{1}{1!} = e(I_0 - I_1)$$
$$\frac{1}{2!} = e(I_1 - I_2)$$
$$\frac{1}{3!} = e(I_2 - I_3)$$
$$\frac{1}{4!} = e(I_3 - I_4)$$

$\therefore 1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24} = e(I_0 - I_1 + I_1 - I_2 + I_2 - I_3 + I_3 - I_4)$

$$\frac{24+12+4+1}{24} = e(I_0 - I_4)$$

$$I_0 - \frac{41}{24e} = I_4$$

$$\therefore I_4 = 1 - \frac{1}{e} - \frac{41}{24e}$$
$$= 1 - \frac{65}{24e}$$

$$I_0 = \frac{1}{0!}\int_0^1 x^0 e^{-x}\,dx$$
$$= \left[-e^{-x}\right]_0^1$$
$$= -e^{-1}+e^0$$
$$= 1 - \frac{1}{e}$$

Student Number: SOLUTIONS

ASCHAM SCHOOL

YEAR 12 Trial Mathematics Extension 2 Exam

MULTIPLE-CHOICE ANSWER SHEET

	A	B	C	D
1.	○	●	○	○
2.	○	○	○	●
3.	○	○	○	●
4.	○	○	○	●
5.	○	○	●	○
6.	○	○	○	●
7.	●	○	○	○
8.	○	○	○	●
9.	○	○	●	○
10.	○	○	○	●